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Understanding Geometric Pattern and its Geometry Part 5 – Patterns on tessellations with regular tiles

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Abstract

We discuss selected aspects of geometric patterns created on tessellations with regular and convex polygons only. We demonstrate how one can use such tessellations for designing a variety of geometric patterns. Various approaches are discussed. The whole discussion is based on historical geometric patterns from Muslim communities known as gereh designs.

Introduction and preliminaries

This paper is a continuation of our discussion on understanding geometric patterns and their geometry (see [6] , [7] , [8] and [9]). Most of its content is closely connected to the high school mathematics curriculum: regular polygons, geometric constructions, and the use of geometry software (Geometer's Sketchpad or GeoGebra).

In this paper, we deal with edge-to-edge tessellations of regular polygons only. We assume that all polygons discussed here are regular and convex, e.g., equilateral triangle, square, regular[1] hexagon, regular octagon, decagon, etc. However, for the sake of simplicity, we will often skip the term 'regular'. We also assume that all regular polygons discussed here are convex. The regular star polygons are not considered here. Thus by saying, for example, an 'octagon,' we mean a regular convex octagon.

Terms 'tessellation' and 'tiling' mean the same.

Lines of symmetry of any object will be called mirrors. Points where two or more mirrors cross will be called kaleidoscopes (compare [3]). Mirror lines will usually be drawn using dashed lines. Edges of tessellation tiles will be drawn using red lines, pattern lines will always be shown using black lines.

All patterns considered here were created according to the gereh rules (see [5] or [6]). Star patterns are only part of this discussion.

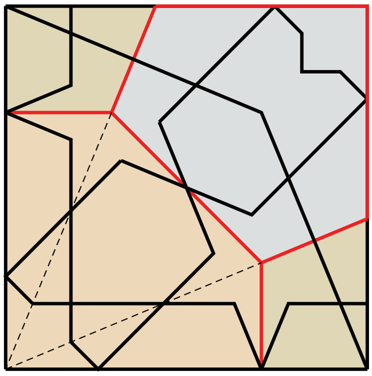
Relation of geometric patterns to their geometric structure

Most of the geometric patterns can be created using a tessellation of symmetric polygons. If a tessellation contains a regular polygon with n edges, we may have a pattern with D_n local symmetry. This is usually a star, a rosette, or even a more complex motif that is symmetric about n of its mirrors. If we have a tessellation without regular polygons, we frequently get a seamless pattern without stars. In the majority of traditional geometric patterns, we usually have a rectangular or triangular fragment that can be used to assemble the whole pattern. This fragment we call a template or a module or, in western terminology, 'a repeat unit'. In most examples with a rectangular template, the edges of the template cross tessellation tiles along one of their symmetry lines. A whole pattern can be reconstructed by reflections of the

[1] Instead of using frequently term 'regular …' we could use names introduced by Tom Ruen: triare, tetrare, pentare, hexare, heptare, octare, etc. (see [13]). However these names are still not widely used.

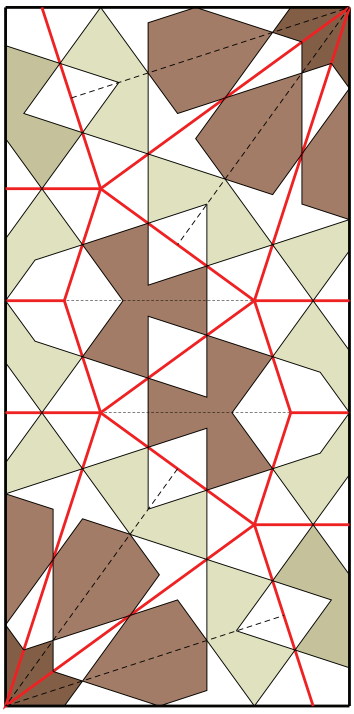
template about its edges or translations about vectors colinear with the template edges. In the case of a triangular template, translations about some vectors are the frequent method for creating a pattern.

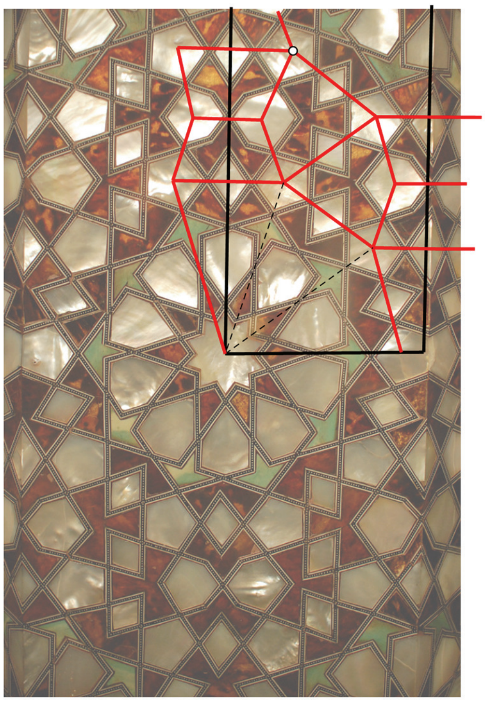
The next drawings show a few examples of patterns and demonstrate the relation of a pattern with its tessellation and edges of the template.



A template of an octagonal pattern

In the left-bottom corner, we have a quarter of a regular octagon. Thus in the resulting pattern (right), we have a star and a rosette with 8 vertices. Each tessellation tile lies completely inside the template or is crossed by the contour along its mirror line.



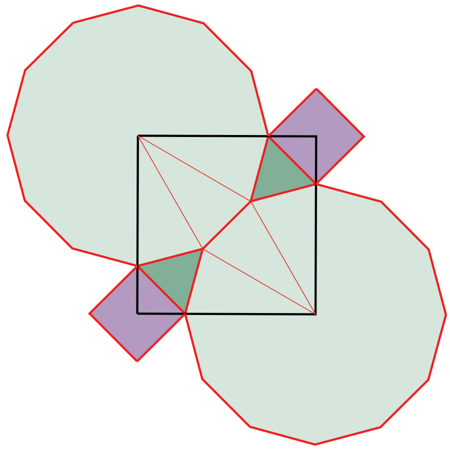


A template of a decagonal pattern (left) and a fragment of a pattern using this template (right)

We have here a regular decagon. Thus we could create a rosette with 10 petals and inside a star with also 10 vertices. Each of the tessellation tiles has at least one mirror. Centers of stars are kaleidoscopes with 10 mirrors passing through them.

As we have noticed in the enclosed examples, many patterns may have a tessellation with one or more regular polygons. In such a case, one can ask if we can produce a pattern using a tessellation with regular polygons only. There are many such examples, and it is worth discussing their properties.

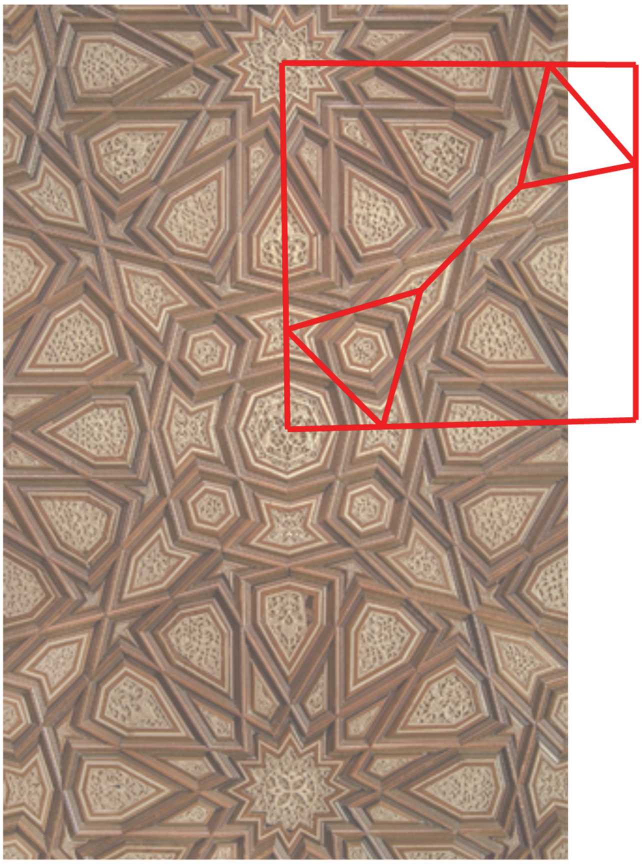
Let us start with an example of a geometric pattern created on a tessellation with regular polygons only.



Fragment of a tessellation with regular polygons and a pattern created using this tessellation

This tessellation can be easily done by starting with a square (contour) and dividing two opposite angles into 3 equal parts.

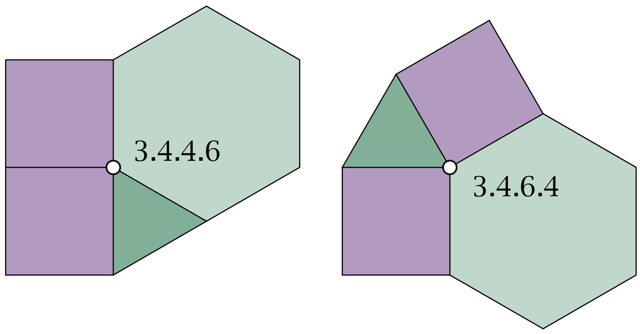
Construction of the pattern was started by inscribing a regular hexagon into the green triangle. The remaining parts of the pattern are a natural consequence of this choice.



Tessellations with regular polygons only were frequently used by artists from all major cultures in our world. We can find them in geometric patterns in Muslim countries, Japanese Kumiko art, Chinese lattice designs, and Byzantian and Roman mosaics. In mathematics, tessellations using regular polygons are part of geometry since Kepler time. In modern geometry, we have plentiful works presenting enormous research of many mathematicians – professionals and amateurs. Worth reading are papers by Darah Chavey (see [1] and [2]). Especially his catalog of tessellations can be a valuable repository of tessellations for creating geometric patterns. Here we can also find a classification of tessellations using regular polygons depending on the types of vertices used in them.

Types of vertices in tessellations with regular polygons

In any tessellation with regular polygons, each vertex is a common point of a few regular polygons. Thus we can assign to it a sequence of natural numbers representing the number of edges of each polygon. For example, a vertex where a regular hexagon, two squares, and one equilateral triangle meet can be represented by a sequence 3.4.4.6. If we want to show the order of connected there polygons, we may use two such sequences, 3.4.4.6 or 3.4.6.4. The next drawing shows these two possible situations.



Two different types of vertices for possible connection of one regular hexagon, two squares, and one equilateral triangle

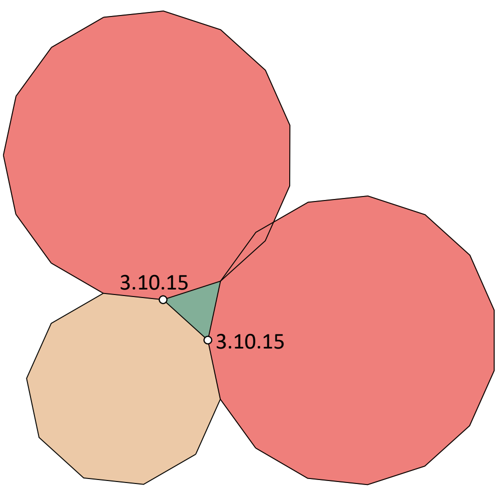
It is easy to notice that these four polygons can meet in one point in two ways only.

It was proved that:

1. *There are only 21 types of vertices possible with regular polygons* (see [4]). All these types are shown in [4] and many other publications. But only some of them may exist in a tessellation with regular polygons.
2. *There is no tessellation using regular polygons with at least one occurrence of vertices of the following types:* 3.7.42, 3.8.24, 3.9.18, 3.10.15, 4.5.20, 5.5.10.
3. *Vertices of the following types* 3.3.4.12, 3.4.3.12, 3.3.6.6, 3.6.3.6, 3.4.4.6, *and* 3.4.6.4 *may occur only in tessellations with at least two different types of vertices.*
4. *For each of the remaining 11 types, there exists a tessellation with all vertices of the given type.*

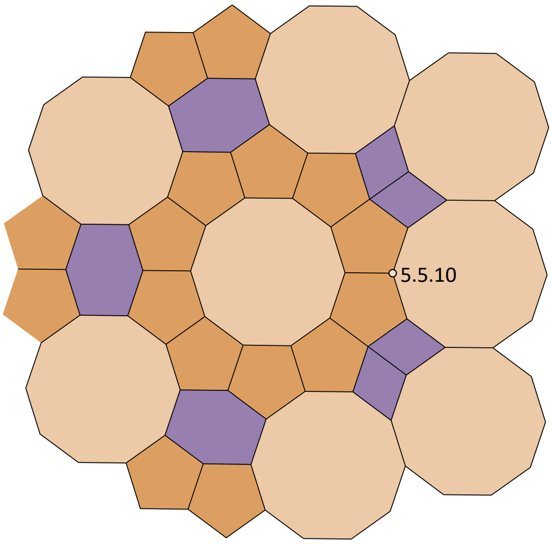
The following table shows all 21 types of possible vertices.

Vertex type	Vertex type	Vertex type	*Comments*
3.3.3.3.3.3	3.3.3.3.6	3.3.3.4.4 3.3.4.3.4	Symbol (+) means that there is no tessellation with regular polygons using this type of vertex. However, such vertex still may exist (see the drawing below). Symbol (*) means that the given vertex may occur only in tessellations with two or more vertex types. For all remaining types of vertices, we can create a tessellation with this type of vertex only. Such tessellations we call Archimedean tessellations.
3.3.4.12 (*) 3.4.3.12 (*)	3.3.6.6 (*) 3.6.3.6	3.4.4.6 (*) 3.4.6.4	
3.7.42 (+)	3.8.24 (+)	3.9.18 (+)	
3.10.15 (+)	3.12.12	4.4.4.4	
4.5.20 (+)	4.6.12	4.8.8	
5.5.10 (+)	6.6.6		



Example for the vertex 3.10.15

The drawing shows what will happen while trying to tessellate the plane with an equilateral triangle, a decagon, and a 15-gon regular polygon. As we can see, there are two vertices with types 3.10.15, but when trying to create a tessellation around them, some of the polygons start overlapping. We could try some other configurations of regular polygons, but we still get similar results.



Example for the vertex 5.5.10

Here we see what may happen if we try to develop a tessellation using vertex type 5.5.10. After building a ring of pentagons around the central decagon, we start obtaining gaps. Here we have two types of gaps. One of them is a rhombus which is not a regular polygon, and another one is a barrel-like hexagon that again is not a regular polygon.

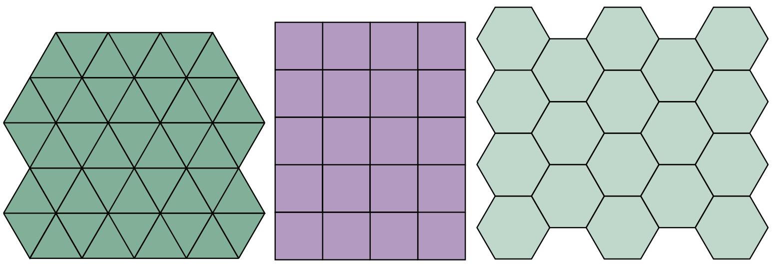
Interestingly, both figures, the rhombus, and the hexagon, are polygons with equal edges and two different angles occurring in a specific order. These two figures are often referred to as shields or semi-regular polygons. We will discuss them in another paper.

Important conclusion: a tessellation with regular polygons may contain equilateral triangles, squares, and regular: hexagons, octagons, and dodecagons only. No other regular polygon can be used in tessellation with regular polygons only (check vertices with (+)).

Tessellations with regular polygons and exactly one vertex type

One of a few ways of classifying tessellations with regular polygons is by the number of types of vertices occurring in the tessellation. Thus we can consider tessellations with one type of vertices, the so-called Archimedean tessellations, tessellations with two different types of vertices, with three different types of vertices, etc.

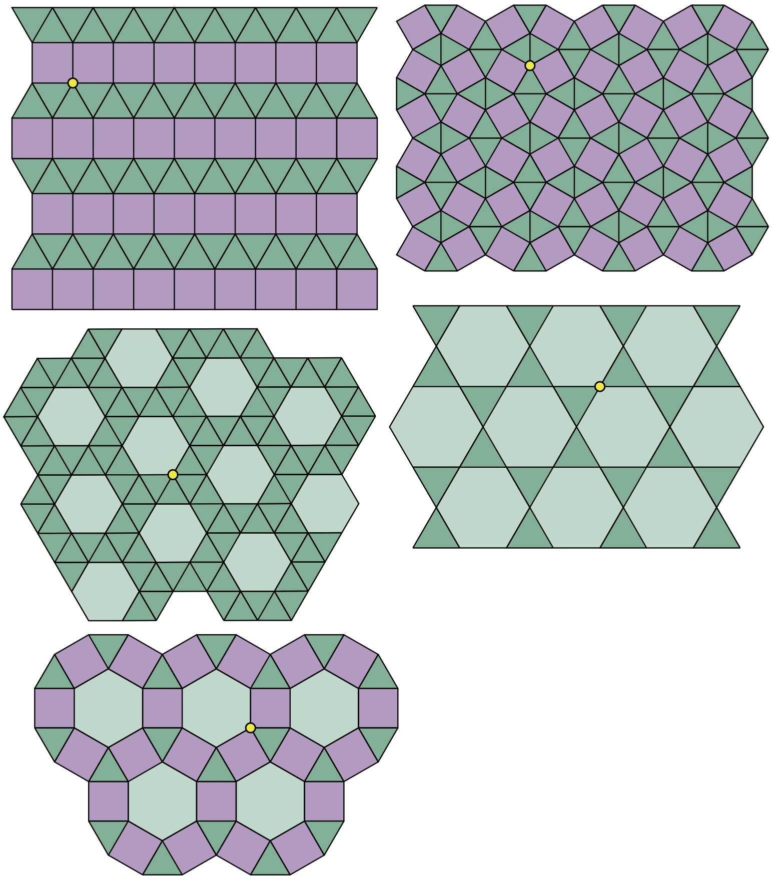
The following drawings show the 11 types of tessellations with regular polygons and one vertex type only.



Tessellations with vertices (from the left)

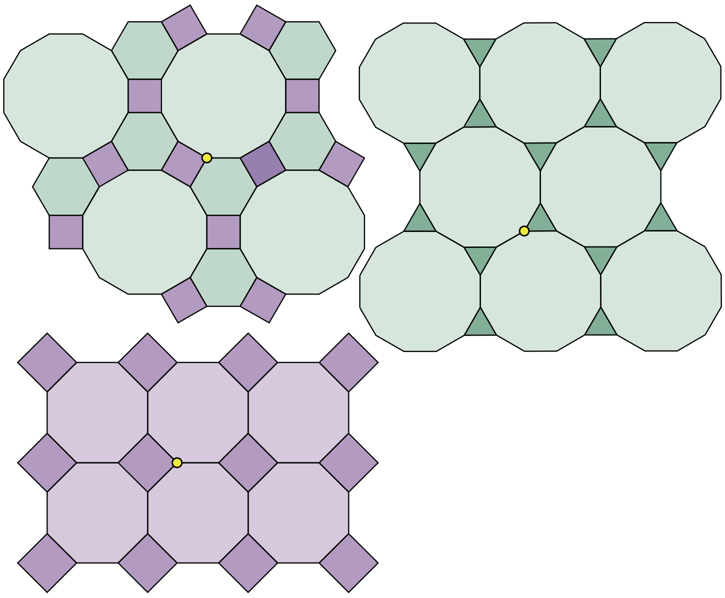
3.3.3.3.3.3 (or 3^6), 4.4.4.4 (or 4^4) and 6.6.6 (or 6^3).

These three tessellations are often called Platonic tessellations.



Tessellations with vertices 3.3.3.4.4, 3.3.4.3.4, 3.3.3.3.6, 3.6.3.6, and 3.4.6.4

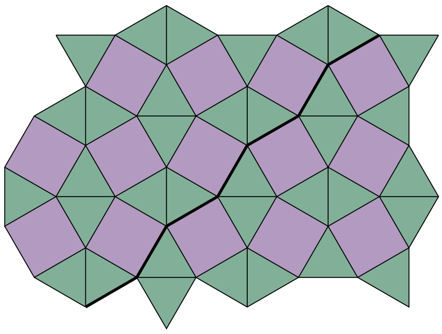
Some of these tessellations can also be seen in Kepler's works.

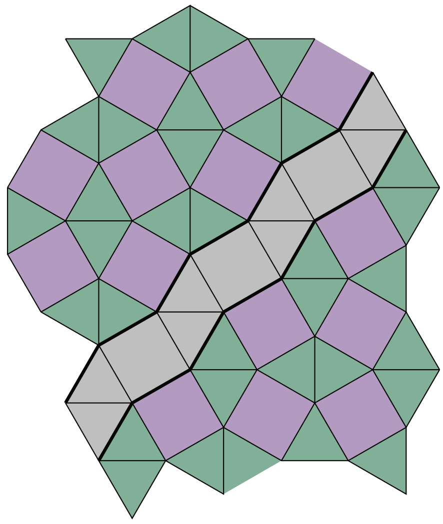


Tessellations with regular dodecagons and regular octagons

Here we have types of vertices 4.6.12, 3.12.12, and 4.8.8.

Important note – we talk here about 11 Archimedean types of tessellations. The word 'types' is essential. This means that we have an infinite number of tessellations representing some of the mentioned here Archimedean tessellation types. Here is one example demonstrating this fact.





Expanding a tessellation

The tiling (3.3.4.3.4) can be split along the zigzag line shown here (above), and in the gap, we can insert a sequence of squares and triangles shown in the right, and then both parts can be merged together (right drawing).

The new tessellation will have vertices of types 3.3.4.3.4 or 3.3.3.4.4 that are considered as the same species (see [4] page 61). Since we can insert a belt of triangles and squares in many places, this leads us to conclude that infinitely many tessellations can be created with an equilateral triangle and square.

The drawings on the previous page allow us to make a conclusion that all tessellations with regular polygons can be split into four groups:

1. Tessellations with one type of a regular polygon (the three Platonic tessellations)
2. Tessellations with triangles, squares, and hexagons
3. Tessellations with a regular dodecagon and other figures mentioned in 2.
4. Tessellations with octagons and squares.

Thus in terms of geometric patterns, we may consider hexagonal patterns, octagonal patterns, and dodecagonal patterns. Each of these groups has several interesting simple patterns as well as many complex structural designs.

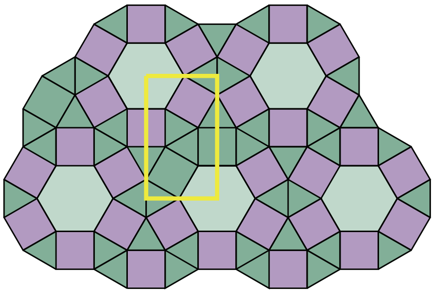
Until now, we were talking about tessellations with one type of vertices. But we could also consider tessellations with vertices of two different types, three and more. So let n denotes the number of types of vertices used in a tessellation, and $T(n)$ is the number of types of tessellations with exactly t different types of vertices. Then we have the following values of the function $T(n)$:

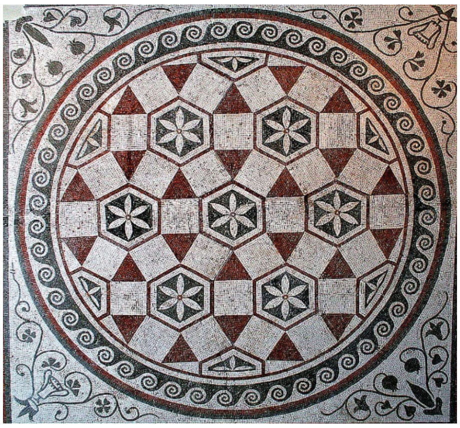
$$T(n) = \begin{cases} 11 & if\ n = 1 \\ 20 & if\ n = 2 \\ 61 & if\ n = 3 \\ 151 & if\ n = 4 \\ 332 & if\ n = 5 \\ 673 & if\ n = 6 \\ ??? & \end{cases}$$

Values of the function $T(n)$ for n>6 are still unknown. For a pattern designer, a very important conclusion comes here – we have infinitely many tessellations with regular polygons that can be used to create geometric patterns. A detailed discussion of this fact can be found in [1] .

Examples of tessellations with regular polygons used in traditional patterns

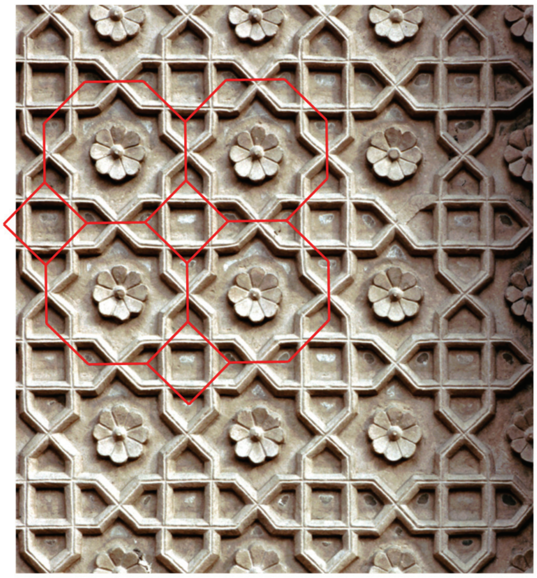
Although there exist infinitely many tessellations with regular polygons, only some were used in traditional works of art. The reason is that some patterns on some of these tessellations are lacking an esthetic appeal. In addition, some of these tessellations are difficult to construct or too complex to be used in an artwork. In the following drawings, we show only a few examples of tessellations frequently used in pattern design.





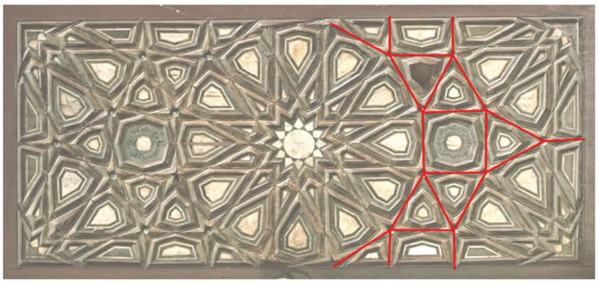
Two examples of tessellations frequently used in Roman art

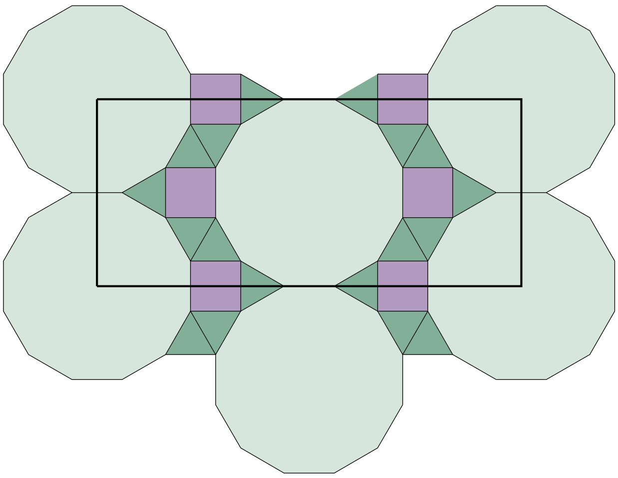
Note, a tessellation can also be treated as a pattern (right image).



Geometric pattern from Agra Fort, India

This pattern was created on a tessellation of regular octagons and squares. Tessellation with regular octagons and squares is the only one in the octagonal group of tilings with regular polygons. However, this tessellation is used as a geometric structure for a large group of patterns. It is also frequently used as a framework for very complex designs.



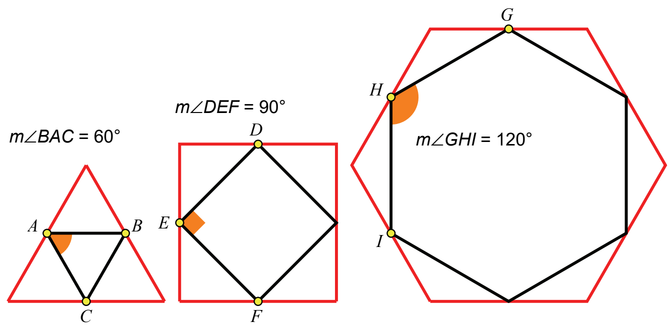


Geometric pattern from Cairo and a tessellation for this pattern

Dodecagonal diversity

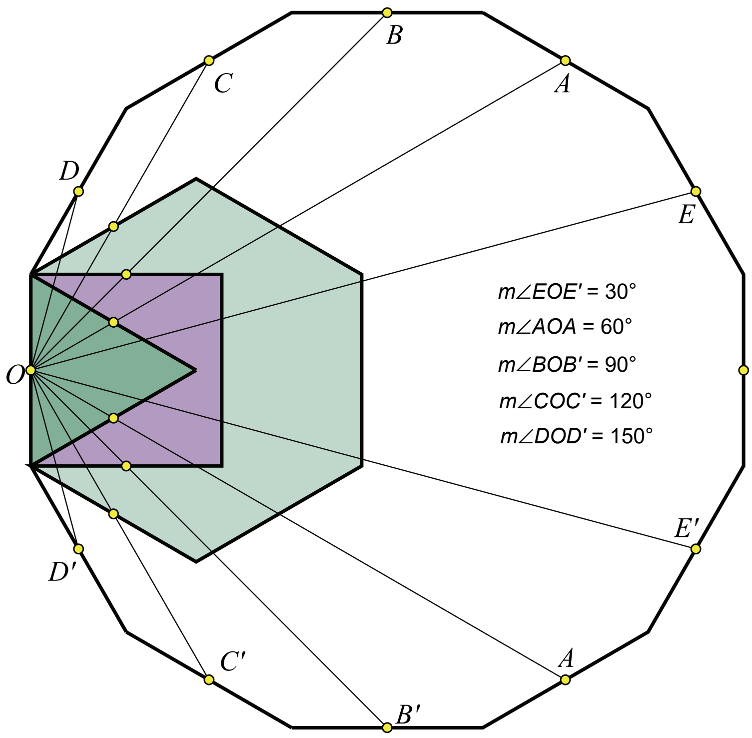
Designing patterns on tessellations with regular polygons can be done in the same way as on decagonal tessellations (see [6] [7]). For the sake of simplicity, we can divide all tessellations with regular polygons into two groups – octagon and square tessellation, and tessellations with regular triangles, squares, hexagons, and dodecagons. This second group could be divided into smaller groups, e.g., tessellations with equilateral triangles and squares. However, it will be more convenient for us to consider them as one group.

Similarly, as with decagonal patterns (see [7]), we could consider patterns where the first line is drawn in various places of the edge of a regular polygon. However, this leads to a huge discussion that may not fit in a journal publication. Thus we will consider only patterns with lines passing through centers of edges of each polygon used in a tessellation. The following drawing shows how one can start a pattern on a particular regular polygon.



Starting a pattern design for various regular polygons

These three examples show that we get different angles by connecting centers of neighboring edges of selected regular polygons. This drawing can be extended onto a regular dodecagon and the variety of angles we could produce.



By connecting midpoints of different edges of a regular dodecagon, we get the same angles as those in the previous figure, plus some new ones.

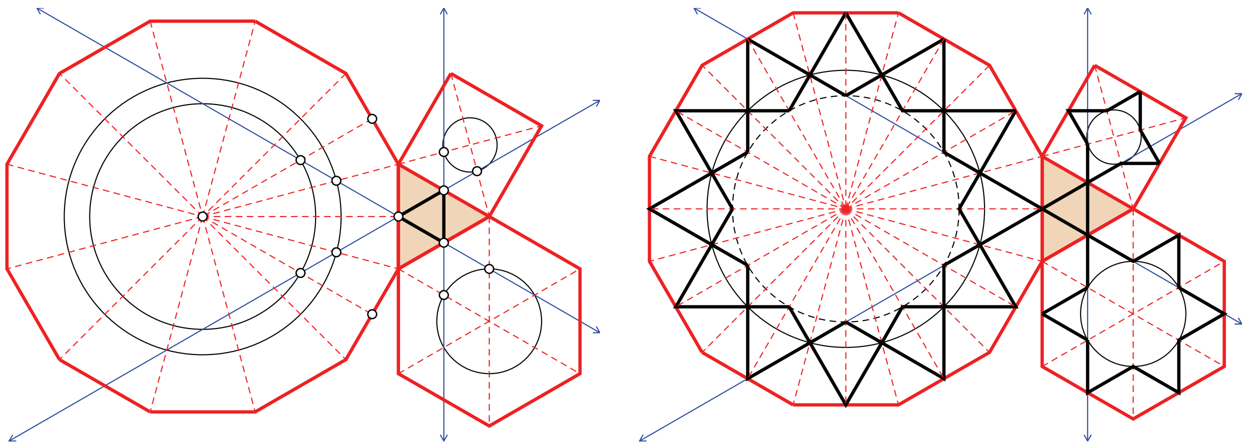
Each of these angles yields patterns with different geometry. Thus we will talk about geometries R#, e.g., R30, R60, R90, etc. Here R stands for regular polygons; the following number is for the angle used in the pattern.

Each of these geometries is connected with an appropriate regular polygon: R60 – triangle, R90 – square, R120 – hexagon, R150 – dodecagon, and R30 – again dodecagon. In general, # can be any angle between 0 and 180 degrees.

Patterns using geometries with wide angles were frequently used in Mamluk designs from Cairo. This way, they were able to produce large dodecagonal rosettes. Narrow angles can be used to create various types of stars.

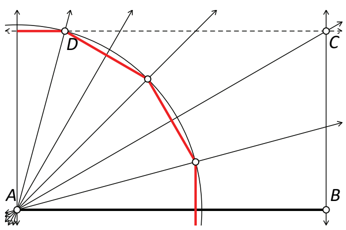
In the next few examples, we will show how some of these geometries can be used in creating geometric patterns. The number of examples presented here is very limited. But, one can take any of the tessellations with regular polygons from [1] and, using presented here ideas develop his own designs.

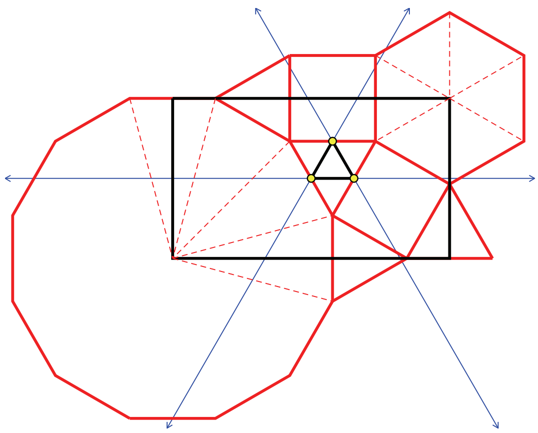
Geometry R60 (determined by the regular triangle)

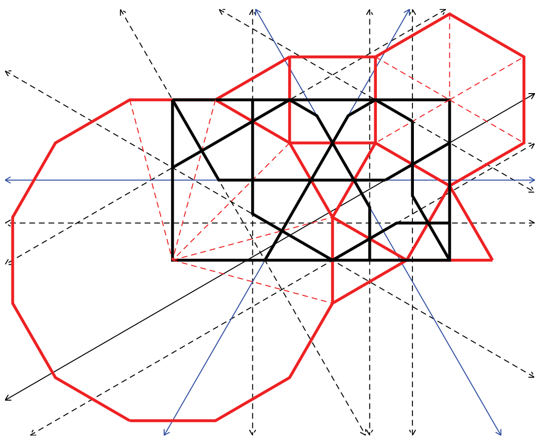


In this pattern geometry, we start creating patterns by drawing the first line through midpoints of the triangle edges (left). This is enough to build a pattern for each of the other shown here polygons. Of course, patterns in the regular dodecagon and regular hexagon may have many variations.

Example for geometry R60



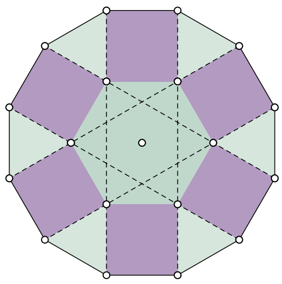


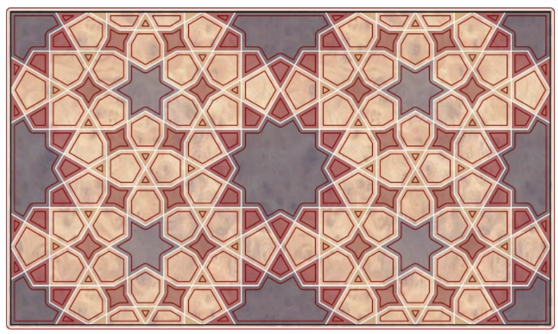


A tessellation

This is a very simple tessellation based on two regular tangent dodecagons. The one on the right-top side was further split into squares, triangles, and a hexagon, as is shown below.

The top-left drawing shows how we start the construction of the tessellation. For segment AB, we draw two perpendicular lines and divide the left-right angle into 6 equal parts. Point c is the point where we draw the top edge of the contour, point d is the first vertex of the dodecagon.

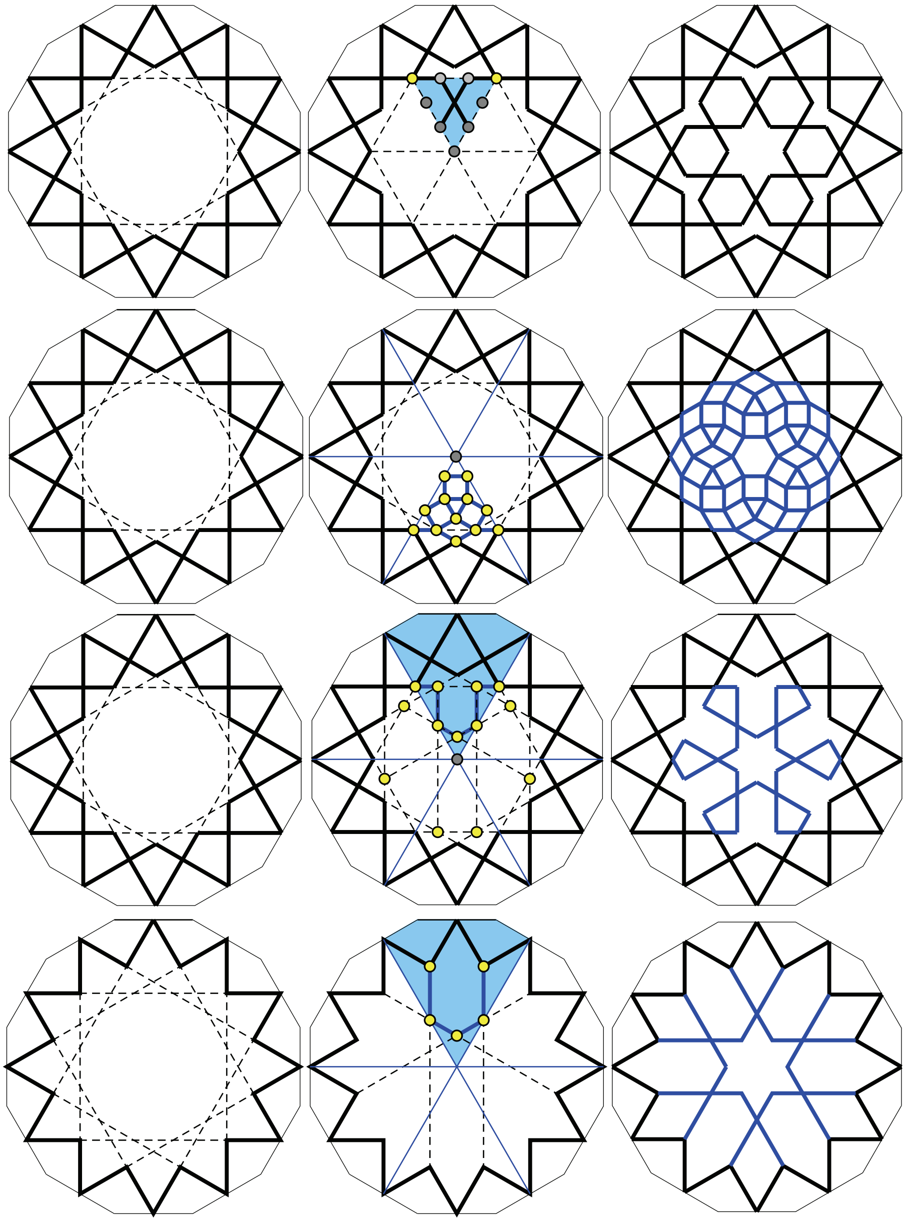




The final pattern

Creation of the pattern

We start designing the pattern by drawing lines through midpoints of the triangle located in the center of the tessellation. All other lines (dashed) are a consequence of this choice.



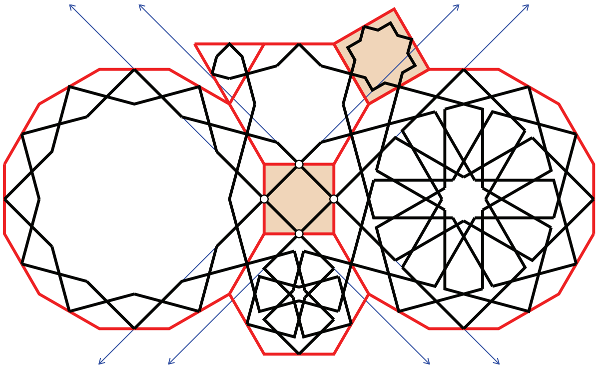
Variations of the pattern for the regular dodecagon

A regular dodecagon in the geometry R60 produces a large star with an open center. In many old works, we can find some interesting attempts to fill this space with extra decorations.

In the left column, we start with a typical dodecagonal star in geometry R60. Then, the central column of drawings shows how to construct the extra decoration. Finally, the right column shows the final result.

Examples presented here were taken from the book by D'Avennes – Islamic Art in Cairo. The last example we can find in the Great Mosque in Axaray, Turkey.

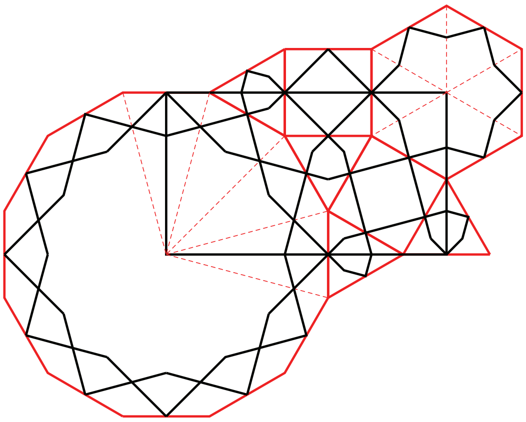
Geometry R90 (determined by the square)

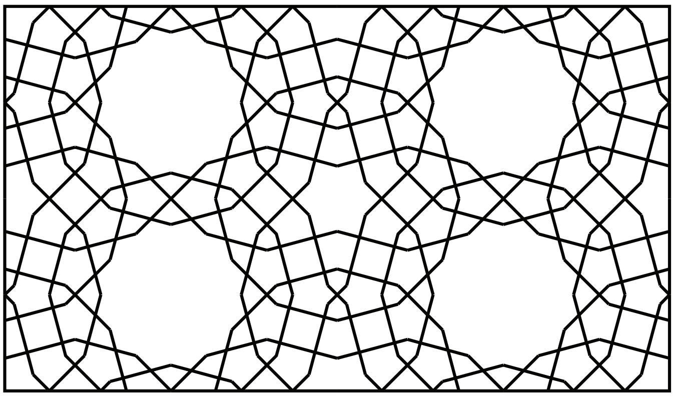


Geometry R90 is determined by the square

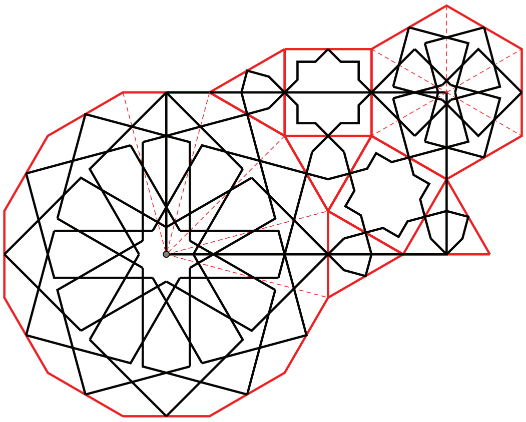
This geometry gives us more ways of creating a pattern. For example, here, we could create a large star with a very large empty center. This space can be filled with an extra rosette. The regular hexagon can also be filled in a few different ways. Here we show only two of them.

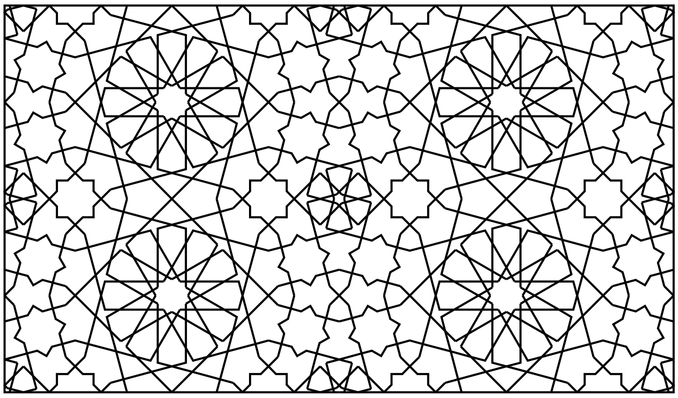
Example for geometry R90





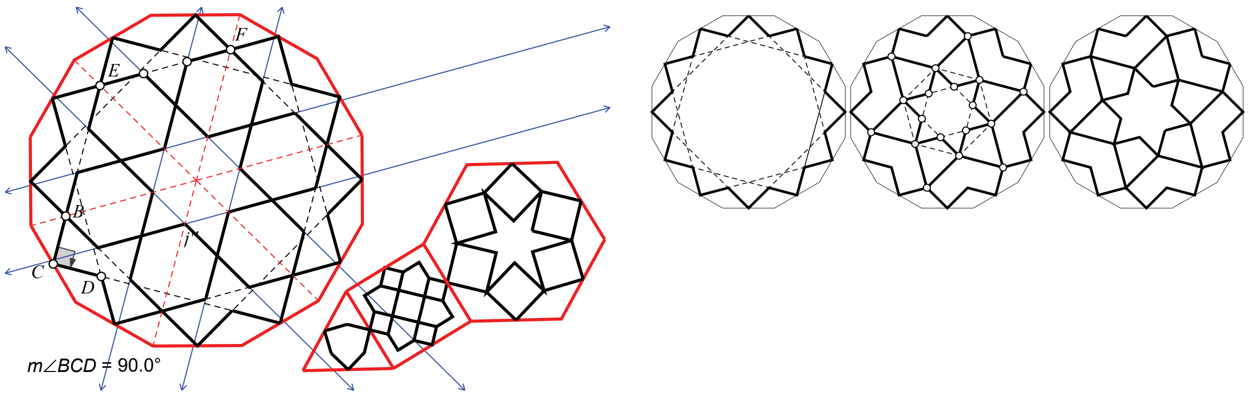
Tessellation from the previous example and a sample pattern created on this tessellation using geometry R90





Tessellation from the previous example and another pattern created on this tessellation using geometry R90

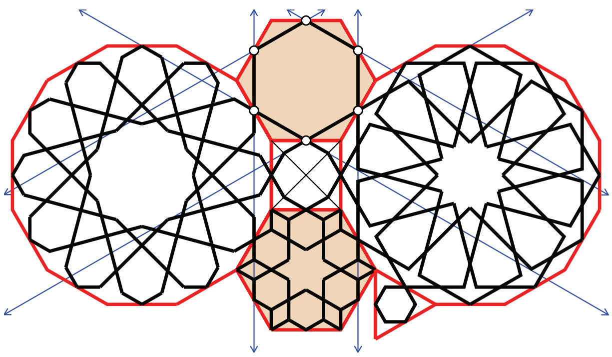
There are a few other ways to fill the regular dodecagon with a pattern compatible with the geometry R90. Two of them are shown below.



Two examples of an extra pattern for the regular dodecagon

Both examples are taken from the al-Salih Tala'i Mosque in Cairo.

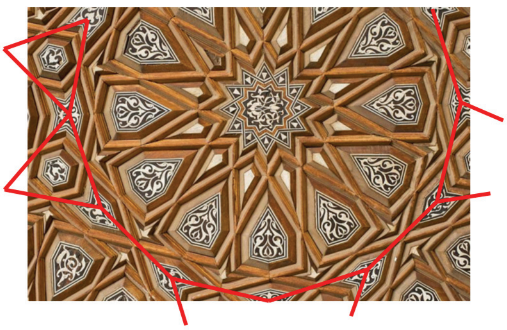
Geometry R120 (determined by the regular hexagon)



Geometry R120

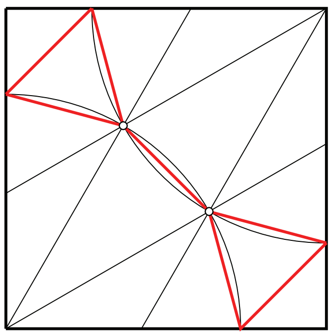
In this case, the hexagon decides about the pattern for all other polygons. This drawing shows two of many ways to fill with a pattern, both regular hexagon, and regular dodecagon.

Example of a pattern using geometry R120



Fragment of a replica of the pattern from the doors from Al Rifa'i Mosque in Cairo

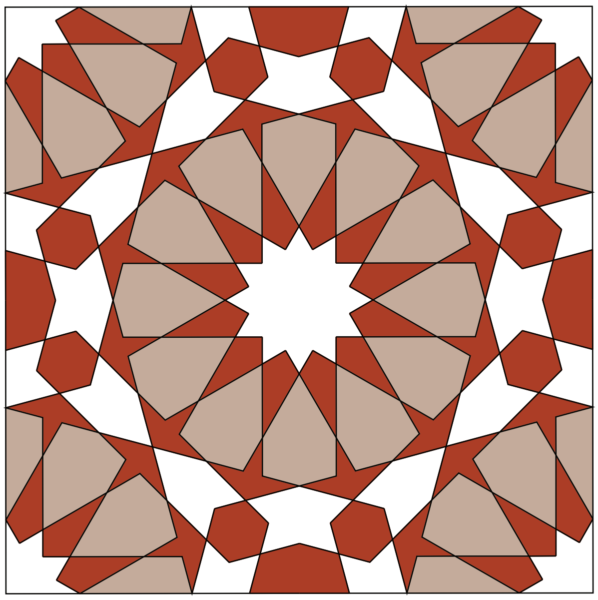
The red lines mark the possible tessellation – two tangent dodecagons with squares and triangles filling the space between them. This is one of many similar designs using geometry C.

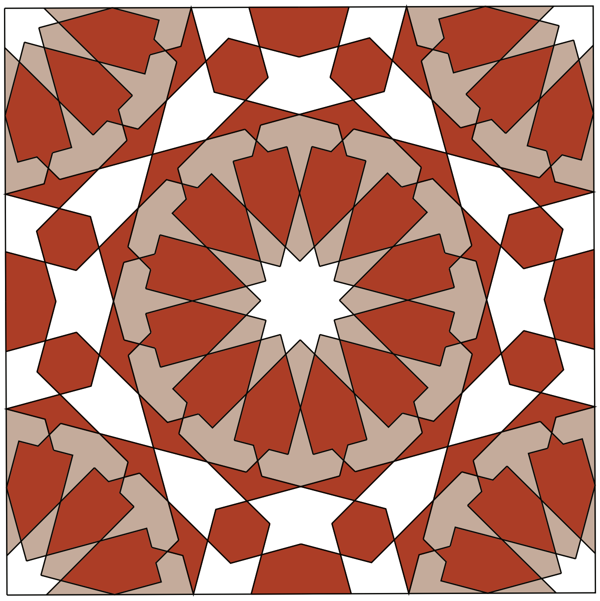


Tessellation for the pattern from Al Rifa'i

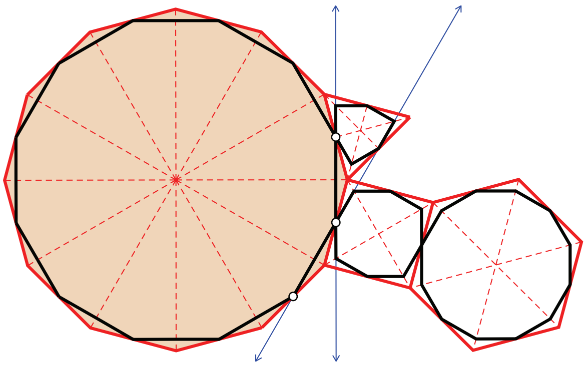
The construction of this tessellation is very simple. Start with a square, divide the two opposite angles into three equal parts and draw arcs through the points shown here.

Below – two versions of the pattern from Al Rifa'i. The right one uses a specific rosette, sometimes referred to as a hammerhead rosette. It frequently occurs in the Maghreb.





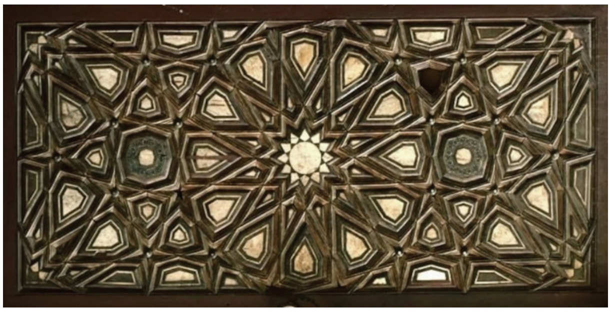
Geometry R150 (determined by the regular dodecagon)



Geometry R150

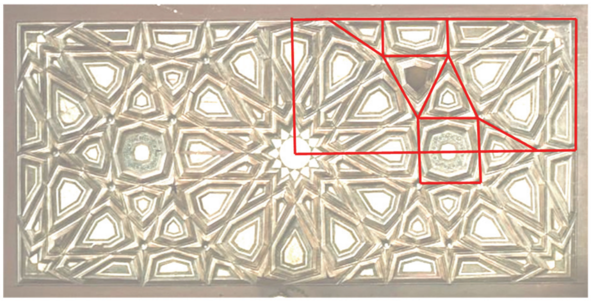
Despite the very wide angle used in this pattern geometry, many interesting designs use this wide angle. The empty space inside of each polygon can be filled with patterns in many interesting ways. One of them is shown in the next example.

Example of a pattern using geometry R150



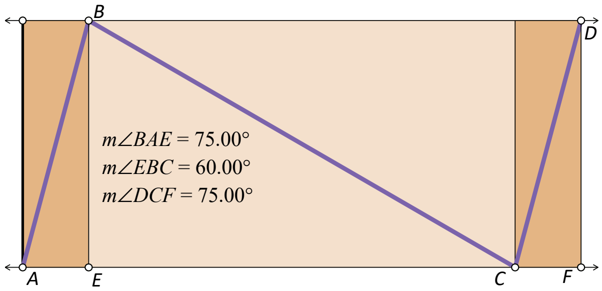
A pattern from Cairo

This is one of many designs that we can find in mosques in Cairo. It is a wooden structure known as kundekari, and it is usually placed on minbars, doors, and window shutters.

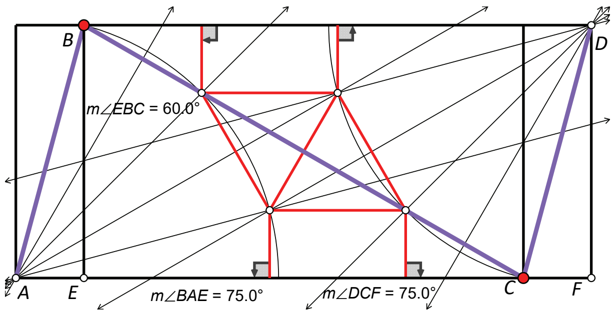


Tessellation for the pattern from Cairo

The red lines show one of a few tessellations that can be used to create this design.

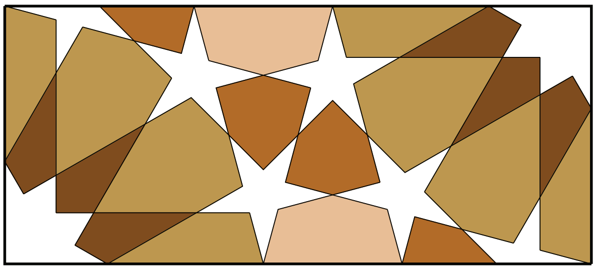


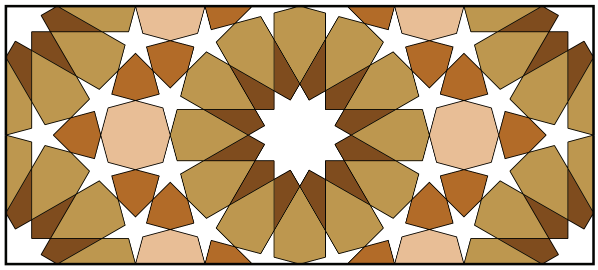
The contour was created as a combination of three rectangles with the bottom edges AE, EC, and CF. Start by drawing the left vertical segment and then add the three rectangles.



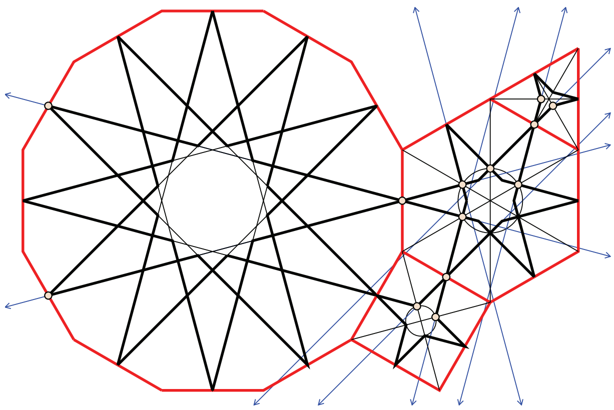
The red points were used to create two arcs. This way, we got quarters of the two dodecagons. Finally, the remaining elements of the tessellation were constructed by joining appropriate vertices of dodecagons and drawing lines perpendicular to the top and bottom edges.

Below we show a template obtained from this tessellation using R150 geometry and a pattern made out of four template copies.





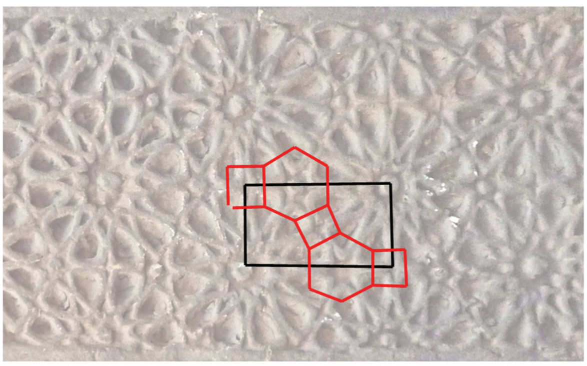
Geometry R30



Geometry R30

With the 30 degrees angle, it is quite difficult to construct geometric patterns. Thus the patterns with R30 geometry are very rare.

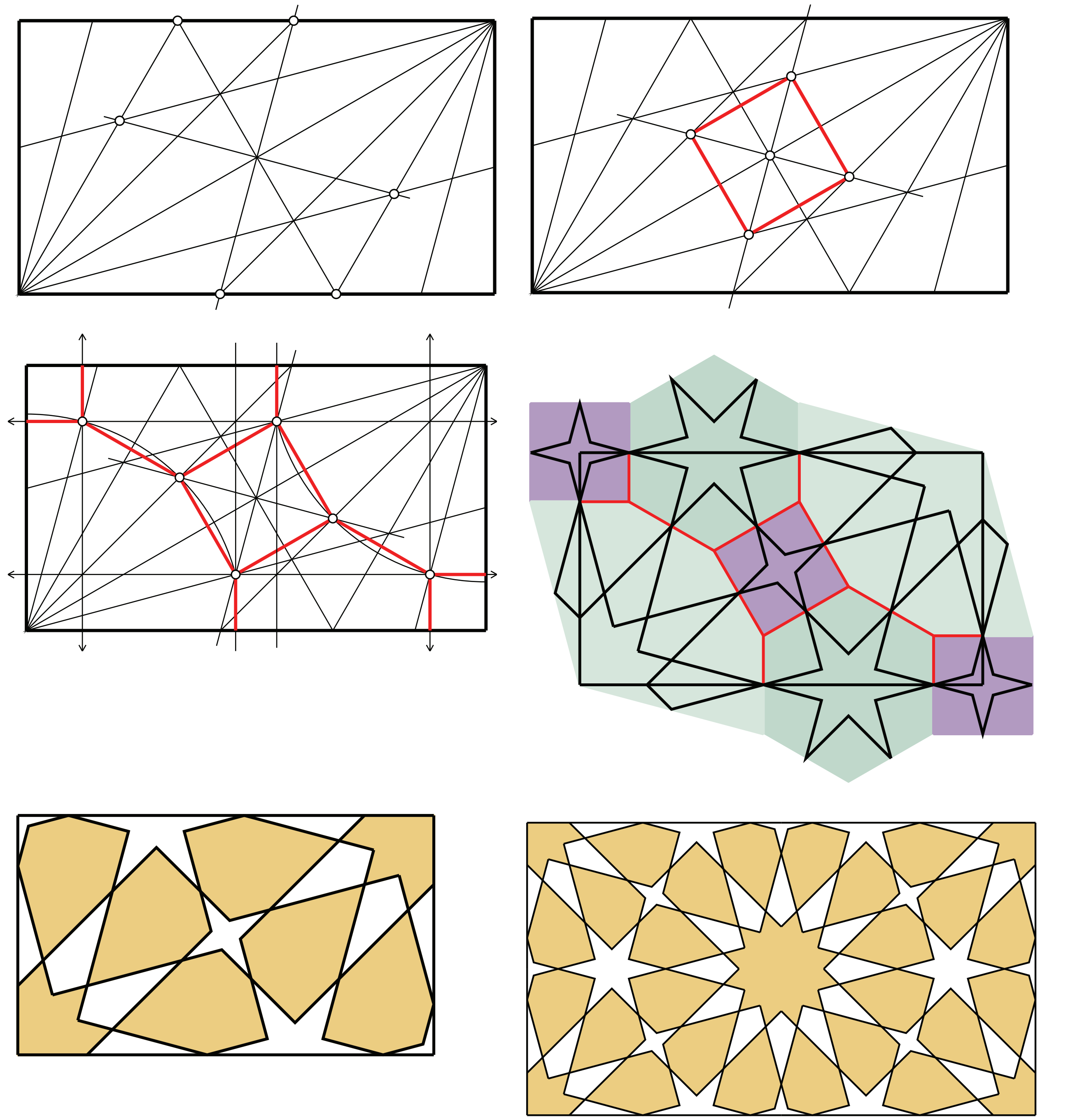
We get here stars with very sharp angles and often ugly, not symmetric shapes between them.



Example of a pattern of type R30

The photo shows a fragment of a pattern carved in plaster from an unknown place in Egypt. The tessellation for it, shown here in red, uses dodecagons, hexagons, and squares.

The following sequence of drawings shows a construction of the contour, tessellation, template for a pattern, and a simple pattern created using four copies of the template.

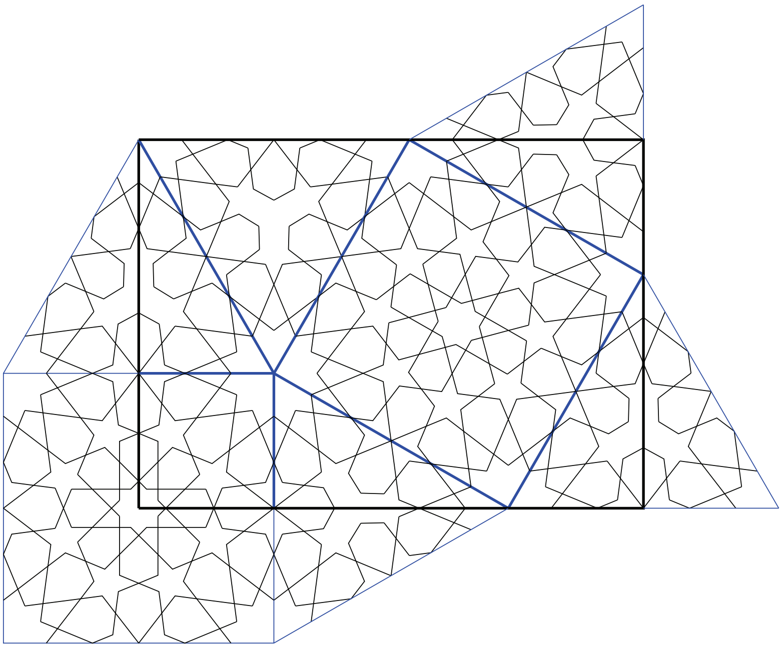


This discussion of various types of patterns created using tessellations with regular polygons only does not show all possible options and variations. But it shows the overall concept. Following this concept, we can use any tessellation with regular polygons only, any angle #, and develop a pattern – existing already or a new one. We can also consider patterns with multiple lines crossing edges of tiles or passing through vertices of tessellation tiles. This can be a topic for a large book on forms of patterns on tessellations using regular polygons.

In this discussion, we did not discuss patterns on tessellations with octagons and squares. This is another topic that can be expanded into a large publication. Therefore, we will leave this topic for another opportunity.

Tessellations with regular polygons as a structure for complex patterns

A tessellation with regular polygons was often used as a framework for a more complex design.



Pattern from the Topkapi scroll

This is one of the patterns from the Topkapi scroll (pattern 35 page 305). It uses squares and equilateral triangles as an overall structure. There are many designs of this type. Note, designs inside of the square and triangle do not use regular polygons.

This particular design can be easily expanded onto a regular hexagon. However, expanding it onto a regular dodecagon is a more difficult task.

Summary

In this paper, we briefly examined selected ways of creating patterns on tessellations with regular polygons. The topic can be a good resource of examples and ideas for teaching school geometry on any level of education. Furthermore, using a geometry tool can make all experiments with such patterns easy and enjoyable for students.

One possible approach is to construct a puzzle with predefined patterns printed on regular polygons and use such a puzzle to experiment with tessellations of regular polygons and design patterns. Although this method was not confirmed by any historical source as a way of creating patterns, it gives us a good opportunity to build some experience in pattern design.

Some attempts of such puzzles can be accessed online through the following links:

http://mirek.majewscy.net/wsp/wsp_octagonal/ (patterns on tessellations with octagons and squares)
http://mirek.majewscy.net/wsp/wsp_archimedean_tilings/ (Archimedean tessellations)
http://mirek.majewscy.net/wsp/03_sultan_ahmed/ (*pattern from Sultan Ahmed Mosque, in Istanbul*).

Final comments

Geometric patterns using tessellations with regular polygons only is a very extensive topic. There are numerous examples of real patterns using such tessellations. In fact, this is a topic for a series of papers or even a complete book. The situation gets even more interesting if we start considering tessellations with regular and semi-regular polygons. This way, we get into a very reach world of designs. Many of them are historical designs that still can be seen in the old architecture of Central Asia, Iran, and Egypt. We may discuss this topic in one of the next papers in this series. This will be possible after publishing a paper on polytiles by Tom Ruen in this or the next issue of the eJMT journal.

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[12] Pattern in Islamic Art: https://patterninislamicart.com

[13] Polytiles page: https://en.wikipedia.org/wiki/User:Tomruen/Polytile

[14] Regular Polygons, https://en.wikipedia.org/wiki/Regular_polygon

[15] Tilings by Regular Convex Polygons: https://en.wikipedia.org/wiki/Euclidean_tilings_by_convex_regular_polygons